JOURNAL OF COMPUTATIONAL PHYSICS **125,** 434–439 (1996)
ARTICLE NO. 0105

Simultaneous Reconstruction of Two Parameters for Transport Equation in a Stratified Half-Space

JINPIN YING, FENG WU, AND WEI SUN

Department of Physics, Zhejiang University, Hangzhou, 310027, China

Received March 6, 1995; revised August 16, 1995

Here, we consider a planar light pulse obliquely incident on the surface of a half-space. A hybrid method that combines the optimization approach with the Green function technique is used to solve the inverse problem of the transport equation. The structure of the fundamental solution is analyzed and the equation for the Green function is derived, together with the initial and boundary conditions. The initial value of the Green function is related to the parameters of the medium and the incident angle. When the boundary data are propagated to the initial value at a certain layer for each incident angle, an objective function is introduced. By using an iterative algorithm (conjugate gradient method), the scattering coefficients and the absorption coefficients are then reconstructed simultaneously layer by layer. The reflection data are required as the input data in the reconstruction algorithm. A detailed algorithm for the reconstruction is presented. Numerical examples are given.

1. INTRODUCTION

The Green function technique has been used to reconstruct one parameter of transport equation [1]. However, simultaneous reconstruction of two parameters is needed in practical application. The recent development and application of various optimization methods have proved its utility as an efficient tool to obtain numerical solutions to inverse problems (see [2–4]). In [4], the optimization approach is used to solve the acoustic inverse problem in the time domain. By introducing a dual function, an explicit expression for the gradient of the objective functional is derived. However, in order to reconstruct the parameters of the acoustic equation, one should solve two sets of PDE equations many times. Thus the algorithm will cost a great deal of CPU time to get the numerical results.

In the present paper, we use a hybrid method that combines the Green function technique with the optimization approach to reconstruct two parameters for the transport equation.

As described in [1], by analyzing the structure of the fundamental solution (i.e., the impulsive response), the time-domain Green function approach has been used to solve the inverse problem of the transport equation in the stratified half-space case. One of the characteristics of the Green function approach is that explicit expression of the initial value of the Green function is related to the medium parameters, and this provides an efficient way to reconstruct the parameters.

In order to reconstruct two parameters of the transport equation simultaneously, the optimization approach is used at each discrete layer. To apply the optimization approach to simultaneous reconstruction, a suitable objective function is introduced first, and then the gradient of the objective function is computed. The gradient of the function is the direction that the function decreases fastest. Once the gradient of the objective function is solved, one can use a conventional steepest descent method, or a conjugate gradient method to minimize the objective function in an iterative way.

In the present paper, using the explicit expression of the initial value of the Green function, we can get the explicit expression of the gradient of the objective function so that we do not need to compute the gradient numerically. In the mean time, we use the characteristic line of transport equation to propagate the boundary value to the initial value layer by layer and take the optimization computation for each layer. The algorithm no longer needs to solve the system of equations of the direct problem as many times as in [4], so the computation of the simultaneous reconstruction is very fast.

The paper is organized as follows. The time-domain transport inverse problem is formulated in Section 2. A Green function technique is used to analyze the fundamental solution and the explicit expression of the initial value is available in Section 3. In Section 4, an objective function is introduced and the numerical algorithm of the optimization approach is presented. The numerical examples are given in Section 5.

2. PROBLEM FORMULATION

Consider such a model: There is a planar light pulse obliquely incident on the surface of a stratified half-space (cf. Fig. 1).

0021-9991/96 $18.00

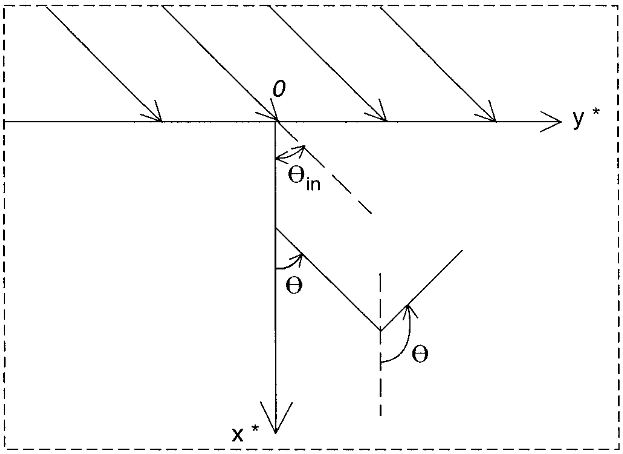

FIG. 1. The scattering configuration.

The intensity $I(x^*, y^*, \theta, t^*)$ ($x^* > 0$) can be modeled by the following equation [1, 5]:

$$\left(\frac{1}{c(x^*)}\partial_{t^*} + \mu\partial_{x^*} + \gamma\partial_{y^*}\right) I(x^*, y^*, \theta, t^*)$$
$$= -[\kappa(x^*) + \sigma(x^*)]\rho(x^*)I(x^*, y^*, \theta, t^*)$$
$$+ \frac{\sigma(x^*)\rho(x^*)}{2\pi} SI(x^*, y^*, \theta, t^*), \tag{1}$$

where

$$\mu = \cos\theta \tag{1.1}$$
$$\gamma = \sin\theta \tag{1.2}$$
$$SI(x^*, y^*, \theta, t^*) = \int_{-\pi/2}^{3\pi/2} p(x^*, \theta, \theta')I(x^*, y^*, \theta', t^*)\, d\theta' \tag{1.3}$$

and $c(x^*)$, $\rho(x^*)$, $\kappa(x^*)$, and $\sigma(x^*)$ are the propagation velocity, density, absorption coefficient, and scattering coefficient, respectively; θ is the angle between the direction of the energy flux and the x^*-axis; $p(x^*, \theta, \theta')$ is the phase function related to the scattering contribution from θ direction to θ' direction. The phase function for scattering $p(x^*, \theta, \theta')$ represents the albedo times the probability density function that a photon is scattered from direction θ into direction θ'. It depends only on the angle $|\theta - \theta'|$ due to the assumption that the scatters are randomly distributed over the stratified half-space [6].

The equation is simplified somewhat by transforming coordinate to

$$x = \int_0^{x^*} \frac{dx'}{c(x')}, \quad t = t^* - \beta\frac{y^*}{c_0}, \quad \beta = \sin\theta_{in},$$

where θ_{in} is the incident angle of the pulse. The shift in the new time variable allows y^* to be eliminated, since the only y^* dependence in I is due to a time delay: given a fixed incident angle θ_{in} with direction cosines $\alpha = \cos\theta_{in}$ and $\beta = \sin\theta_{in}$, the response of the medium to an input is independent of y^*, but must be time-shifted because the input does not reach every point on the boundary at the same time. Specifically, the intensity may be written as

$$I(x^*, y^*, \theta, t^*) = I\left(x^*, \theta, t^* - \beta\frac{y^*}{c_0}\right) = I(x, \theta, t).$$

The intensity equation now has the form

$$\left[\left(1 - \beta\gamma\frac{c}{c_0}\right)\partial_t + \mu\partial_x\right] I(x, \theta, t)$$
$$= -(\kappa + \sigma)\rho c I(x, \theta, t) + \frac{\sigma\rho c}{2\pi} SI(x, \theta, t),$$

If c is not a constant, a second coordinate transformation can be used to straighten the characteristics. However, for the sake of simplicity, the assumption of $c = c_0$ is considered. The transport equation of an obliquely incident pulse reduces to a simpler form,

$$[\omega\partial_t + \mu\partial_x]I(x, \theta, t) = -b(x)I(x, \theta, t) + q(x)SI(x, \theta, t). \tag{2}$$

where

$$\omega = 1 - \beta\gamma = 1 - \sin\theta_{in}\sin\theta \tag{2.1}$$
$$b(x) = [\kappa(x^*) + \sigma(x^*)]\rho(x^*)c_0 \tag{2.2}$$
$$q(x) = \sigma(x^*)\rho(x^*)c_0/2\pi. \tag{2.3}$$

It is assumed that the incident light pulse reaches the surface $x = 0$ at the time $t = 0$, and, thus, causality gives

$$I(x, \theta, t) = 0, \quad t < \alpha x.$$

The boundary conditions are

$$I(x = 0, \theta, t) = I^i(t)\delta(\theta - \theta_{in}), \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \tag{4}$$

$$I(x = 0, \theta, t) = R(\theta, t), \quad \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \tag{5}$$

where $I^i(t)$ is the time-dependent incident energy intensity, and $R(\theta, t)$ is the intensity of the reflected flux at the surface $x = 0$ in the direction θ, $\theta \in (\pi/2, 3\pi/2)$. Note that $R(\theta, t)$ is a measurable quantity at the surface $x = 0$ in the inverse problem and will be used as input data in the simultaneous reconstruction of the scattering and absorption coefficients in the medium $x > 0$.

3. GREEN FUNCTION APPROACH

In this section, we consider the structure of the fundamental solution. The fundamental solution $\tilde{I}$ is the solution to Eq. (2) due to an impulsive (δ-function) incidence at the surface; i.e., it satisfies

$$[\omega\partial_t + \mu\partial_x]\tilde{I}(x, \theta, t) = -b(x)\tilde{I}(x, \theta, t) + q(x)S\tilde{I}(x, \theta, t) \tag{6}$$

$$\tilde{I}(x = 0, \theta, t) = \delta(t)\delta(\theta - \theta_{in}), \quad \theta_{in} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \tag{7}$$

The solution to the transport equation (2) for an incidence with an arbitrary time-dependence $I^i(t)$ is then immediately obtained from the following time convolution integral

$$I(x, \theta, t) = \int_0^t \tilde{I}(x, \theta, t')I^i(t - t')\, dt'. \tag{8}$$

Removing the singular part from the fundamental solution $\tilde{I}$ and denoting the regular part by G (which will be referred as the Green function hereafter), one has

$$\tilde{I}(x, \theta, t) = T(x, \theta)\delta(t - \alpha x)\delta(\theta - \theta_{in}) + G(x, \theta, t)H(t - \alpha x), \tag{9}$$

where $H(t - \alpha x)$ is the Heaviside step function vanishing for $t < \alpha x$; $T(x, \theta)$ will be determined below. Substituting Eq. (9) into Eq. (6) and matching the coefficients of the $\delta'(t - \alpha x)\, \delta(\theta - \theta_{\text{in}})$, $\delta(t - \alpha x)\, \delta(\theta - \theta_{\text{in}})$, $\delta(t - \alpha x)$, and $H(t - \alpha x)$ terms, respectively, one has

$$\mu\partial_x T(x, \theta) + b(x)T(x, \theta) = 0, \tag{10}$$

$$(\omega - \mu\alpha)G(x, \theta, t = \alpha x) - q(x)S[T(x, \theta)\delta(\theta - \theta_{in})] = 0, \tag{11}$$

$$[\omega\partial_t + \mu\partial_x]G(x, \theta, t) = -b(x)G(x, \theta, t) + q(x)SG(x, \theta, t), \tag{12}$$

From Eq. (10), one obtains (cf. Eq. (7))

$$T(x, \theta) = \exp\left[-\frac{1}{|\mu|}\int_0^x b(x')\, dx'\right]. \tag{13}$$

Equation (11) gives

$$G(x, \theta, t = \alpha x) = \frac{q(x)p(x, \theta, \theta_{in})T(x, \theta_{in})}{\omega - \alpha\mu}, \quad \theta \neq \theta_{in} \tag{14}$$

Similarly, one can obtain the following initial value of the Green function from Eq. (12) when $\theta = \theta_{\text{in}}$ [7]:

$$G(x, \theta_{\text{in}}, t = \alpha x) = T(x, \theta_{\text{in}}) \left\{\exp\left[\frac{1}{\alpha}\int_0^x q(x')p(x', \theta_{\text{in}}, \theta_{\text{in}})\, dx'\right] - 1\right\}. \tag{15}$$

Note that the initial condition (14) and the transport equation (12) of the Green function $G(x, \theta, t)$ will be used for the reconstruction of the parameters in the inverse transport problem.

Equation (3) indicates that

$$G(x, \theta, t) = 0, \quad t < \alpha x. \tag{16}$$

In the inverse transport problem, the boundary values of the Green function at the surface $x = 0$ are given as the input data, i.e.,

$$G(x = 0, \theta, t) = \begin{cases} r(\theta, t), & \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \\ 0, & \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \end{cases} \tag{17}$$

where $r(\theta, t)$ is related to $R(\theta, t)$ through a time convolution integral

$$R(\theta, t) = \int_0^t r(\theta, t')I^i(t - t')\, dt'. \tag{18}$$

Note that if one uses an incident light pulse with an arbitrary time-dependent $I^i(t)$, then $r(\theta, t)$ should be obtained from the above equation, i.e., through a deconvolution.

4. OPTIMIZATION APPROACH

In the present paper, we want to use a hybrid method that combines the optimization approach and the Green function technique to reconstruct the scattering and absorption coefficient simultaneously. However, we find it more instructive to give the optimization approach a mathematical description first.

The objective function $J(x_1, x_2)$ which will be introduced in Section 5 is

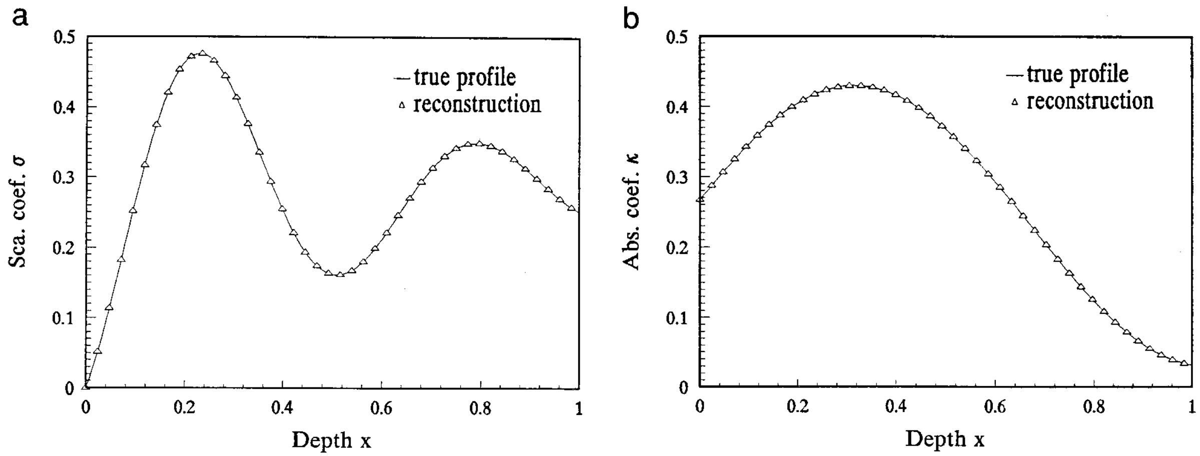

FIG. 2. The simultaneous reconstruction of scattering and absorption coefficient using 256 gridpoints (smooth profile).

$$J(x_1, x_2) = \sum_{i=1}^{m} [x_1\eta_i \exp(\xi_i x_2) - \lambda_i]^2. \quad (19)$$

The problem of simultaneous reconstruction is to find a group of x_1, x_2 which minimizes $J(x_1, x_2)$. The minimum of $J(x_1, x_2)$ is zero if the inverse problem has a solution.

To apply the optimization approach to an inverse problem, one sets a suitable objective function first and then computes the gradient of the objective function. The gradient of the function is the direction that the function increases fastest. Once the gradient of the objective function is computed, one can use a conventional steepest descent method, or a conjugate gradient method to minimize the objective function in an iterative way. In the present paper, we construct such an objective function, as above, that one has an explicit expression of the gradient

$$\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} 2\eta_i \exp(\xi_i x_2)[x_1\eta_i \exp(\xi_i x_2) - \lambda_i] \\ \sum_{i=1}^{m} 2\eta_i\xi_i \exp(\xi_i x_2)x_1[x_1\eta_i \exp(\xi_i x_2) - \lambda_i] \end{bmatrix}, \quad (20)$$

where g_1, g_2 are the gradients of the objective function for x_1, x_2, respectively.

In the present paper, we use a standard conjugate gradient method (Polak–Ribiere algorithm [8]) to minimize $J(x_1, x_2)$ and reconstruct x_1, x_2 simultaneously.

Assume that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (21)$$

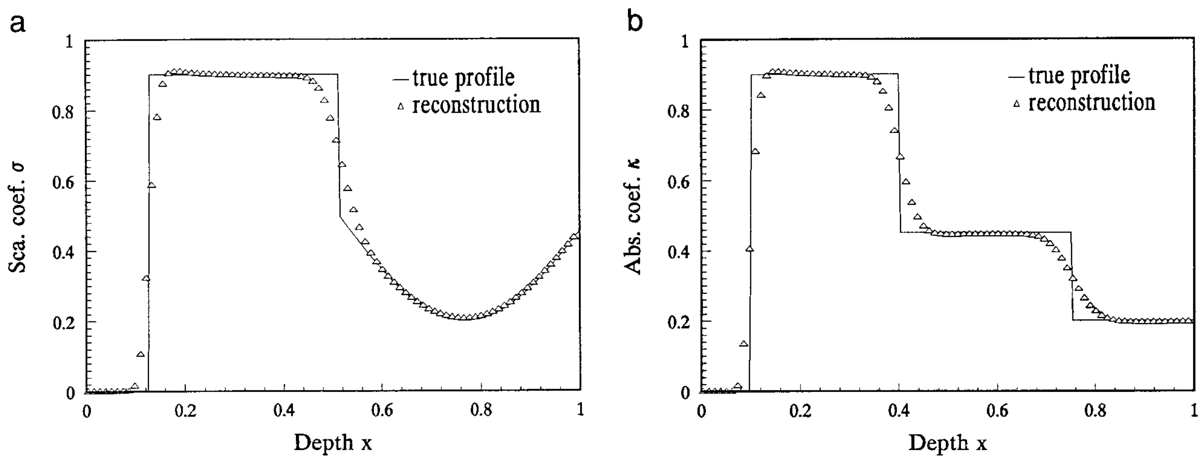

FIG. 3. The simultaneous reconstruction of scattering and absorption coefficient using 256 gridpoints (noncontinuous profile).

The iterative algorithm is described as follows:

Step 0. Select an initial approximation (guess) $\mathbf{x} = \mathbf{x}^0$.

Step 1. Set $i = 0$, $\mathbf{g}^0 = \mathbf{h}^0 = -\mathbf{g}(\mathbf{x}^0)$.

Step 2. Compute a scalar step size $a^i > 0$ by one dimension linear searching:

$$J(\mathbf{x}^i + a^i\mathbf{h}^i) = \min\{J(\mathbf{x}^i + a\mathbf{h}^i) \mid a \geq 0\}. \tag{22}$$

Step 3. Improve the reconstruction by setting

$$\mathbf{x}^{i+1} = \mathbf{x}^i + a^i\mathbf{h}^i. \tag{23}$$

Step 4. Compute the new gradient $\mathbf{g}(\mathbf{x}^{i+1})$.

Step 5. If $\mathbf{g}(\mathbf{x}^{i+1}) = 0$, stop; else, set

$$\mathbf{g}^{i+1} = -\mathbf{g}(\mathbf{x}^{i+1}) \tag{24}$$

$$\mathbf{h}^{i+1} = \mathbf{g}^{i+1} + \gamma^i\mathbf{h}^i \tag{25}$$

with

$$\gamma^i = \frac{\langle \mathbf{g}^{i+1} - \mathbf{g}^i, \mathbf{g}^{i+1}\rangle}{\langle \mathbf{g}^i, \mathbf{g}^i\rangle} \tag{26}$$

and set $i = i + 1$; go to step 2.

If the inverse problem has a unique solution, the numerical reconstruction should converge to the correct solution when the objective function is minimized to zero.

5. RECONSTRUCTION SCHEME AND NUMERICAL IMPLEMENTATION

In the inverse problem, the boundary values for the Green function at surface $x = 0$ are given as the input data:

$$G(x = 0, \theta, t) = \begin{cases} r(\theta, t), & \theta \in \left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right], \\ 0, & \theta \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]. \end{cases} \tag{17}$$

In this section, as a numerical example, we simultaneously reconstruct the parameters $b(x)$ and $q(x)$. Once the $b(x)$ and $q(x)$ are reconstructed, the scattering coefficient $\sigma(x)$ and the absorption coefficient $\kappa(x)$ will be solved when $p(x, \theta, \theta')$ is assumed to be known:

$$b(x) = [\kappa(x^*) + \sigma(x^*)]\rho(x^*)c_0 \tag{27}$$

$$q(x) = \sigma(x^*)\rho(x^*)c_0/2\pi. \tag{28}$$

An algorithm for the simultaneous reconstruction of the two parameters can be described as follows:

(1) Choose two incident angles $\theta_{in}^{(1)}$, $\theta_{in}^{(2)} \in (-\pi/2, \pi/2)$. Get two measurable reflectance distributions $r_1(\theta, t)$, $r_2(\theta, t)$, $\theta \in (\pi/2, 3\pi/2)$, respectively.

(2) Assume that the grid function values $G_1(x_i, \theta, t)$ (for $\theta_{in}^{(1)}$) and $G_2(x_i, \theta, t)$ (for $\theta_{in}^{(2)}$) at a certain stage x_i are known (note that this is always true at the surface $x_i = 0$; cf. Eq. (17) and step (1)). Choose an arbitrary angle $\theta_m \in (\pi/2, 3\pi/2)$ and propagate $G_1(x_i, \theta, t)$, $G_2(x_i, \theta, t)$ to the initial value of the next stage $G_1(x_{i+1}, \theta_m, t = \alpha_1 x_{i+1})$, $G_2(x_{i+1}, \theta_m, t = \alpha_2 x_{i+1})$ by using the algorithm that is described in [1, 7].

(3) We mark $G_1(x_{i+1}, \theta_m, t = \alpha_1 x_{i+1})$, $G_2(x_{i+1}, \theta_m, t = \alpha_2 x_{i+1})$ as G_1, G_2, respectively and define that

$$\begin{aligned}\tilde{G}_1 &= \tilde{G}_1(x_{i+1}, \theta_m, t = \alpha_1 x_{i+1}) \\ &= \frac{q(x_{i+1})p(x_{i+1}, \theta_m, \theta_{in}^{(1)})T(x_{i+1}, \theta_{in}^{(1)})}{\omega_1 - \alpha_1\mu}\end{aligned} \tag{29}$$

$$\begin{aligned}\tilde{G}_2 &= \tilde{G}_2(x_{i+1}, \theta_m, t = \alpha_2 x_{i+1}) \\ &= \frac{q(x_{i+1})p(x_{i+1}, \theta_m, \theta_{in}^{(2)})T(x_{i+1}, \theta_{in}^{(2)})}{\omega_2 - \alpha_2\mu}\end{aligned} \tag{30}$$

for $\theta_{in}^{(1)}$ and $\theta_{in}^{(2)}$ according to Eq. (14), respectively.

Now we introduce the objective function in the least squares form [2]:

$$J(q, b) = J(q(x_{i+1}), b(x_{i+1})) = \sum_{k=1}^{2} (\tilde{G}_k - G_k)^2. \tag{31}$$

Assuming that

$$\begin{aligned}\int_0^{x_{i+1}} b(x')\,dx' &= \int_0^{x_i} b(x)\,dx + \int_{x_i}^{x_{i+1}} b(x)\,dx \\ &\cong \int_0^{x_i} b(x')\,dx' + b(x_{i+1})(x_{i+1} - x_i)\end{aligned} \tag{32}$$

and marking that

$$\eta_k = \frac{p(x_{i+1}, \theta_m, \theta_{in}^{(k)})T(x_i, \theta_{in}^{(k)})}{\omega_k - \alpha_k\mu} \tag{33}$$

$$\xi_k = -\frac{1}{|\alpha_k|}(x_{i+1} - x_i) \tag{34}$$

$$\lambda_k = G_k. \tag{35}$$

From Eqs. (32)–(35), the objective function will then reduce to a simple form

$$J(q, b) = \sum_{k=1}^{2} (q\eta_k \exp(\xi_k b) - \lambda_k). \tag{36}$$

Note that the form of $J(q, b)$ is similar to the form of the objective function discussed in the above section. So with the optimization method described in Section 4, the value $q(x_{i+1}), b(x_{i+1})$ at the stage x_{i+1} can be reconstructed simultaneously.

(4) Once we know the value $q(x_{i+1})$, $b(x_{i+1})$ at the stage x_{i+1}, the values $G_1(x_{i+1}, \theta, t)$, $G_2(x_{i+1}, \theta, t)$ can be calculated by using the finite difference method described in [1, 7].

(5) Set $i = i + 1$, return to step 2, and reconstruct the parameters q, b, layer by layer.

Here, one should note that if we choose more incident angles than two, the inverse problem may be overdetermined, which may give the better numerical reconstruction [4].

As the first numerical example, we assume that $p(x, \theta, \theta') = \sin(\theta - \theta')^2$. The parameters $\sigma(x)$ and $\kappa(x)$ are chosen to be smooth. In Figs. 2a, b, the solid lines represent the true profiles, respectively. Note that all the parameters are scaled to be dimensionless. The parameters $\sigma(x)$ and $\kappa(x)$ then can be reconstructed simultaneously according to the method described above. The simultaneous reconstructions of $\sigma(x)$ and $\mathrm{k}(x)$ using 256 gridpoints are shown in Fig. 2a and Fig. 2b. As shown in figures, the reconstruction errors for smooth profiles are very small and the reconstruction results agree with the true profiles very well.

Because of the error of the finite difference method that is used in the present inverse algorithm, the reconstruction errors become large for the noncontinuous profiles but still agree with the true profiles. As the second numerical example, the parameters $\sigma(x)$ and $\kappa(x)$ are chosen as a step function followed by a smooth function, as shown by the solid line in Figs. 3a, b. The phase function is same as above. The triangle symbols in Figs. 3a, b represent the simultaneous reconstruction of the parameters $\sigma(x)$ and $\kappa(x)$, respectively.

6. CONCLUSION

In the present paper, we use the Green function technique to analyze the fundamental solution of the transport equation, and we get the explicit expression of the initial condition. An objective function is introduced. The explicit expression of the gradient of the objective function has been derived. The parameters are then reconstructed by combining the conjugate gradient method with the characteristics method. The numerical example with good results is given.

ACKNOWLEDGMENT

The authors thank especially Dr. Sailing He (Dept. of Electromagnetic Theory, Royal Institute of Technology, Sweden) for very fruitful discussions.

REFERENCES

1. J. P. Ying and W. Sun, *SPIE* **2393** (1995).
2. A. Bamberger and G. Chavent, *Geophys.* **47,** 757 (1982).
3. M. S. Frank, and C. A. Balanis, *IEEE Trans. Geosci. Remote Sensing* **GE-25,** 691 (1987).
4. S. He, S. I. Kabanikhin, *et al.* Technical Report TRITA-TET 94-X, Royal Institute of Technology, Stockholm, Sweden, June, 1994 (unpublished).
5. R. J. Krueger and R. L. Ochs, *Wave Motion* **11,** 525 (1989).
6. M. J. C. Van Gemert, S. L. Jacques, H. J. C. M. Sterenborg, and W. M. Star, *IEEE Trans. Biomed. Eng.* **36**(12), 1146 (1989).
7. K. L. Krider, *Transp. Theory Statist. Phys.* **18**(5,6), 513 (1989)–(1990).
8. E. Police, *Computational Methods in Optimization, a Unified Approach* (Academic Press, New York, 1971), p. 53.